Exact renormalization in quantum spin chains

Hong-Hao Tu and Mikel Sanz

Max-Planck-Institut für Quantenoptik, Hans-Kopfermann-Str. 1, 85748 Garching, Germany

We introduce a real-space exact renormalization-group method to find exactly solvable quantum spin chains and their ground states. This method allows us to provide a complete list for exact solutions within SU(2) symmetric quantum spin chains with $S \leq 4$ and nearest-neighbor interactions, as well as examples with $S=5$. We obtain two classes of solutions. One of them converges to the fixed points of renormalization group and the ground states are matrix-product states. Another one does not have renormalization fixed points and the ground states are partially ferromagnetic states.

I. INTRODUCTION

Understanding the physical properties of quantum many-body systems is an important common issue in condensed-matter physics and quantum-information theory. The number of parameters required to describe a random state grows exponentially with the number of particles, which makes the computation of many-body systems very difficult, even numerically.[1] However, recent development in quantum-information theory implies that only a corner of such a huge Hilbert space is relevant for describing the low-energy states of physical systems.[2,3] The characteristic feature of this corner seems to be an area law:[4] the von Neumann entropy of a subsystem in the many-body ground state scales with the border area, rather than the volume—the case for a random state. This means that the ground states of quantum many-body systems usually only contain a small amount of entanglement. It is natural to take this advantage and design clever parametrizations of states which both capture the essential physics and allow classical simulations with a polynomial time. In one dimension (1D), the matrix-product state (MPS) (Refs. 5–7) is a candidate for such a purpose. The MPS lies at the heart of the success of the density-matrix renormalization group (DMRG),[8–12] which has been proved to be an accurate numerical method for describing the low-energy states of quantum lattice models. Recently, there are many interesting extensions along this direction, including infinite MPS for critical systems,[13] continuous MPS for quantum field theories,[14] and projected entangled pair state (PEPS) for higher dimensional systems.[15–17]

The MPS also appears to be the exact ground states of certain spin models. For example, the valence-bond solid (VBS) ground states of the Affleck-Kennedy-Lieb-Tasaki (AKLT) models[18] are matrix-product states. They provide a clear physical picture to the Haldane gap phenomena[19] and shed light on their "nearby" integer-spin Heisenberg antiferromagnets.[20] In condensed-matter physics, the Hamiltonians usually arise with two-body interactions and SU(2) symmetry since they are relevant to describe realistic materials. In order to study such systems, a method has been suggested in Ref. 21 to construct the SU(2) symmetric two-body parent Hamiltonians for MPS. However, when starting with the Hamiltonians, in principle, it can be extremely hard to find their matrix-product ground states.[22]

The purpose of this paper is to investigate a real-space renormalization group and its applications in a systematical search for exactly solvable quantum spin chains. The present approach complements the parent Hamiltonian method in Ref. 21, such that one can start from the Hamiltonians and search for exactly solvable ones. We first briefly review the basics of real-space renormalization and its extension to systems with SU(2) symmetry. The presence of symmetry allows us to design a simple exact renormalization scheme. By using this method, we study quantum spin chains with SU(2) symmetry and nearest-neighbor interactions. For $S \leq 4$, we provide complete solutions for the models which are frustration free for two neighboring spins. Moreover, we also provide a MPS solution of $S=5$ which was not previously known. We discuss these exact solutions by dividing them into two different classes, whose ground states are matrix-product states and partially ferromagnetic states, respectively.

II. REAL-SPACE EXACT RENORMALIZATION

Let us consider a chain with N local d-dimensional Hilbert spaces $\mathcal{H}$ that we can assume local spins. We denote by $|M\rangle \in \mathcal{H}$ an orthonormal basis in $\mathcal{H}$. And let us also consider a translationally invariant Hamiltonian $H = \Sigma_i h_i^{(k)}$ containing local interaction terms acting on contiguous k sites. We can assume positive semidefinite interaction $h_i \geq 0$, since they can always be achieved by shifting local-energy level.

Let us now briefly explain the real-space renormalization process. We start by coupling the first two spins, whose Hilbert space $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}_2$ is mapped into a Hilbert space $\mathcal{H}_2$ which has, in general, a dimension $D_2 < d^2$. The criteria followed to perform this reduction is to conserve only the low-energy states of the Hamiltonian. In general, the method works by finding the mappings $\mathcal{A}^{[i]}: \mathcal{H}_{i-1} \otimes \mathcal{H} \to \mathcal{H}_i$ which carry out this process. We continue this renormalization procedure until reaching the end of the chain and getting an orthonormal basis $\{|\chi\rangle\}_{\chi=1}^{D_N}$ of the Hilbert space $\mathcal{H}_N$.

Let us show the real-space renormalization process from the $(i-1)$th spin to the ith spin, which can be written in a basis as[10]

$$|\beta[i]\rangle = \sum_{\alpha, M} A_{\alpha, \beta}^{[M]} |\alpha[i-1]\rangle \otimes |M[i]\rangle, \tag{1}$$

where the input state $|\alpha[i-1]\rangle \in \mathcal{H}_{i-1}$, the output state $|\beta[i]\rangle \in \mathcal{H}_i$, and the Kraus operator $A^{[M_i]}$'s are $D_{i-1} \times D_i$ ma-
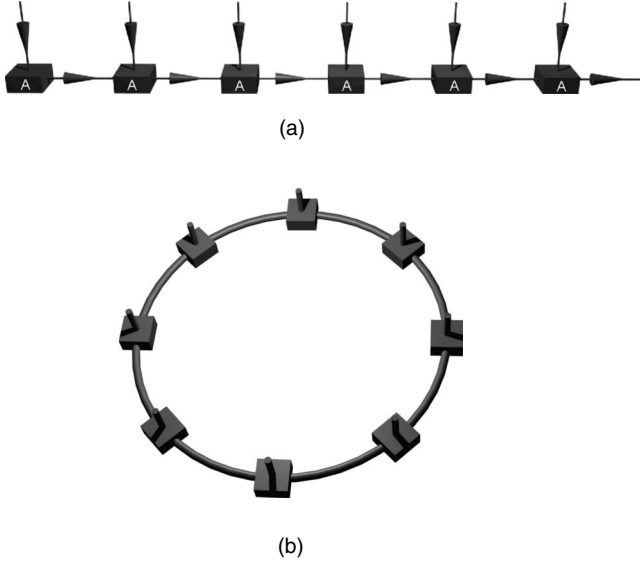
(a)

(b)

FIG. 1. Real-space renormalization. (a) The real-space renormalization group yields matrix-product states. (b) In periodic boundary condition, the translationally invariant MPS are constructed from the fixed-point Kraus operators.

trices satisfying isometry condition $\sum_M A^{[M]\dagger} A^{[M]} = \mathbb{1}$. Here we define $D_0 = 1$ so that the Kraus operator $A^{[M_1]}$ for the first spin can be viewed as a row vector.

Equation (1) shows the real-space renormalization results in an orthonormal basis $|\chi\rangle$ $(\chi = 1 \ldots D_N)$ with a matrix-product form [see Fig. 1(a)]

$$|\chi\rangle = \sum_{M_1 \ldots M_N} (A^{[M_1]} A^{[M_2]} \ldots A^{[M_N]})_\chi |M_1, M_2, \ldots, M_N\rangle,$$

(2)

where $D = \max_i D_i$ is called the *bond dimension* of the matrix product. In the DMRG algorithm, these matrix-product states are used variationally to find the best approximation of the low-energy sector of the 1D systems.

In this work, we are interested in special models such that the states $|\chi\rangle$ *exactly* span the ground-state subspace in the thermodynamic limit. The specifications about the thermodynamic limit comes from the fact that every state for N sites can be written by means of a matrix-product ansatz given in Eq. (2) by taking $D > d^{N/2}$. However, we seek for models for which an exact renormalization can be performed for arbitrarily long chains. In other words, the ground states of these models can be solved rigorously through the real-space renormalization, and the truncation induced by the Kraus operators does not harm.

Practically, since $h_i \geq 0$, this search can be achieved if the Kraus operators for each spin can be adjusted step by step in the renormalization group to fulfill

$$\text{Tr}(\rho_i^\chi h_i) = 0, \quad \forall \ i = 1 \ldots N \quad \text{and} \quad \forall \ \chi = 1 \ldots D_N,$$

(3)

where $\rho_i^\chi = \text{Tr}_{\text{env}}[|\chi\rangle\langle\chi|]$ is the reduced density matrix for k spins. The above condition leads to $H|\chi\rangle = 0$, which means that the vectors $|\chi\rangle$ are the ground states of H, because H

≥ 0. Such Hamiltonians are called *frustration-free Hamiltonians* since their ground states minimize energy locally. For instance, it is well known that the ferromagnetic Heisenberg chain is a typical frustration-free model in which all the spins tends to align in parallel to gain energy. Recently, the frustration-free Hamiltonians have been reformulated as quantum k-SAT problems and attract considerable interests in quantum-information community.[23–26]

For generic models, this renormalization procedure will terminate after blocking a number of spins due to intrinsic frustrations. To find the exactly solvable model, the first possibility is that the renormalization group reaches a fixed point. Then, the ground state of the Hamiltonian in periodic boundary condition can be written as a translationally invariant MPS [see Fig. 1(b)]

$$|\Psi\rangle = \sum_{M_1 \ldots M_N} \text{Tr}(A^{[M_1]} A^{[M_2]} \ldots A^{[M_N]})|M_1, M_2, \ldots, M_N\rangle,$$

where the Kraus operators $A^{[M]}$ are the converged $D \times D$ matrices at the fixed point. We discuss these fixed-point MPS solutions in Sec. III B. Another possibility is that, for some models, the number of states $\dim \mathcal{H}_k$ that we should keep, increases when gathering more spins. Even though there is no renormalization fixed point, we find that it is still possible to obtain the ground states exactly if $\dim \mathcal{H}_k$ increases in a *controllable* way. We illustrate this point in Sec. III C, when discussing the partially ferromagnetic states.

III. QUANTUM SPIN CHAINS WITH SU(2) SYMMETRY

In this section, we adapt the real-space exact renormalization to SU(2) symmetric quantum spin chains with nearest-neighbor interactions. Therefore, let us start by explaining some details about the SU(2) symmetric Hamiltonians. The most general SU(2) symmetric translationally invariant spin-S Hamiltonian with nearest-neighbor interactions can be expressed as

$$H = \sum_i \sum_{n=1}^{2S} a_n (\vec{S}_i \cdot \vec{S}_{i+1})^n + a_0 \mathbb{1}.$$

(4)

The study of these SU(2) symmetric models has a long history in condensed-matter physics. It was known that some of these models can be solved by Bethe Ansatz method and such models are fully classified by solutions of Yang-Baxter equations.[27]

We want to identify the frustration-free models in Eq. (4) and find their ground states through real-space exact renormalization. However, it is convenient to use projectors instead of spin operators, so we use the transformation

$$(\vec{S}_i \cdot \vec{S}_{i+1})^n = \sum_{S_T=0}^{2S} \left[\frac{1}{2} S_T(S_T+1) - S(S+1) \right]^n P_{S_T}(i, i+1),$$

where P_{S_T} is a projector onto total-spin S_T states of the two spins. By shifting the local-energy levels, we can always rewrite Hamiltonian (4) as a sum of projectors
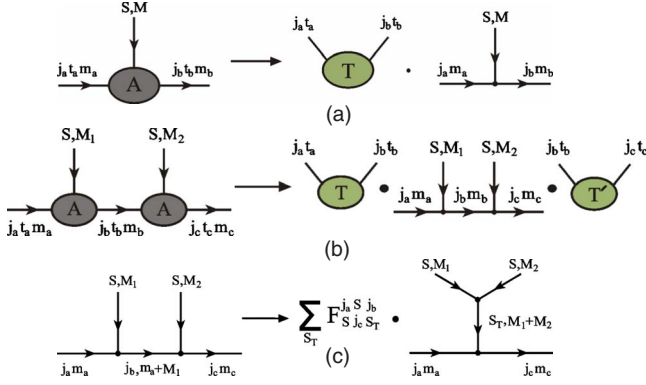
FIG. 2. (Color online) Real-space renormalization with SU(2). (a) An isometry $A^{[S,M]}$ is decomposed as a matrix T and a fusion of angular momenta. (b) The renormalization of two spins can be done in two successive steps. (c) The same input and output states with different intermediate fusion channels for two spins can be related by the F symbol.

$$H = \sum_i \sum_{S_T \in \mathcal{K}} J_{S_T} P_{S_T}(i, i+1) \qquad (5)$$

with coupling constants $J_{S_T} > 0$ and $\mathcal{K} \subseteq [0, 2S]$ is a set specifying the choice of projector(s) as local interactions. Since the local interactions in Eq. (5) are a sum of projectors, we have $H \geq 0$.

Let us remark that, as the physical representation is irreducible and we restrict to nearest-neighbor interactions, the exact value of J_{S_T} is not important whenever the Hamiltonian is frustration free.

From the projector Hamiltonian (5), it is still not clear how to properly choose, if possible, the set $\mathcal{K}$ to make the Hamiltonian frustration free. However, as we restrict ourselves to frustration-free models with two neighboring spins, we can provide a complete list by taking advantage of the renormalization group.

A. Exact renormalization with SU(2) symmetry

In this section, we explain how to make use of the SU(2) symmetry in the exact renormalization scheme. This particularizes the real-space renormalization in Eq. (1) such that both the input and output states form representations of the symmetry group, which ensures the symmetry is preserved in each renormalization step. The method shown here is a three-step process.

Equation (1) can be promoted to an SU(2) adapted basis[28–30]

$$|j_b t_b m_b\rangle = \sum_{j_a t_a m_a} \sum_M A^{[S,M]}_{j_a t_a m_a, j_b t_b m_b} |j_a t_a m_a\rangle |SM\rangle, \qquad (6)$$

where the j's denote the SU(2) representations (total-spin quantum number), the t's distinguish the degenerate states within the same j, and the m's are the magnetic quantum numbers associated with j.

The first step of the process consists of splitting the Kraus operators into two terms by means of Wigner-Eckart theorem [see Fig. 2(a)] as

$$A^{[S,M]}_{j_a t_a m_a, j_b t_b m_b} = T_{j_a t_a, j_b t_b} \langle j_a m_a, SM | j_b m_b\rangle, \qquad (7)$$

where the indices $j_a t_a, j_b t_b$ keep track of the representations of the input and output states. The first term is a real matrix T denoting the weights of different input states in each output states. We call this matrix "weight matrix." Let us remark that the weight matrix does not depend on the magnetic quantum numbers. The second term is the Clebsch-Gordan coefficient $\langle j_a m_a, SM | j_b m_b\rangle$, corresponding to the representation fusion $j_a \otimes S \rightarrow j_b$. To ensure that the output states always form an orthonormal basis, the weight matrix must fulfill

$$T_{j_a t_a j_b t_b} = 0 \quad \text{unless} \quad |j_a - S| \leq j_b \leq j_a + S, \qquad (8)$$

$$\sum_{j_a t_a} T_{j_a t_a j_b t_b} T_{j_a t_a j_b t'_b} = \delta_{t_b, t'_b} \qquad (9)$$

for every j_b. The first constraint is related to SU(2) fusion rules. The second constraint means the columns of $T_{j_a t_a, j_b t_b}$ corresponding to the same j_b but different t_b, are orthonormal vectors, which guarantees the isometry condition $\sum_M A^{[S,M]\dagger} A^{[S,M]} = \mathbb{1}$ for the Kraus operators.

The advantage of this representation for the Kraus operators is that it allows us to design an elegant way to perform the exact renormalization group, which is the second step of the method. Let us consider two neighboring spins [see Fig. 2(b)]. The renormalization process consists of two sequential representation fusions $j_a \otimes S \rightarrow j_b$ and $j_b \otimes S \rightarrow j_c$. As a result, we obtain the orthonormal basis

$$|j_c t_c m_c\rangle = \sum_{M_1 M_2} \sum_{j_a t_a m_a} \sum_{j_b t_b m_b} T_{j_a t_a j_b t_b} \langle j_a m_a, SM_1 | j_b m_b\rangle$$
$$\times \langle j_b m_b, SM_2 | j_c m_c\rangle T'_{j_b t_b, j_c t_c} |j_a t_a m_a\rangle |SM_1\rangle |SM_2\rangle, \qquad (10)$$

where the weight matrices T and T' for these two spins can be different, in general. Alternately, the renormalization process Eq. (10) can be done by fusion of the two physical spins to their coupled representations $S \otimes S \rightarrow S_T$ first and then $j_a \otimes S_T \rightarrow j_c$. In the latter fusion sequence, we obtain the same basis

$$|j_c t_c m_c\rangle = \sum_{j_a t_a m_a} \sum_{S_T M_T} R^{S_T}_{j_a t_a j_c t_c} \langle j_a m_a, S_T M_T | j_c m_c\rangle |j_a t_a m_a\rangle |S_T M_T\rangle, \qquad (11)$$

where $|S_T M_T\rangle$ is the coupled basis of two physical spins. The two different fusion channels are unitarily related by the re-coupling F symbol [see Fig. 2(c)] defined by $F^{j_a S j_b}_{S S_T j_c} = \langle j_b(j_a S), S; j_c m_c | j_a, S_T(SS); j_c m_c\rangle$. By using Wigner's 6-$j$ symbol, this F symbol can be expressed as[31]

$$F^{j_a S j_b}_{S S_T j_c} = (-1)^{j_a + j_c} \sqrt{(2j_b + 1)(2S_T + 1)} \begin{Bmatrix} j_a & S & j_b \\ S & j_c & S_T \end{Bmatrix}.$$

By substituting this in Eq. (10) and comparing with Eq. (11), we obtain

$$R^{S_T}_{j_a t_a, j_c t_c} = \sum_{j_b t_b} T_{j_a t_a, j_b t_b} F^{j_a S j_b}_{SS_T j_c} T'_{j_b t_b, j_c t_c}. \tag{12}$$

According to Eq. (11), the output states $|j_c t_c m_c\rangle$ only keep the local ground states of Hamiltonian (5) if

$$R^{S_T}_{j_a t_a, j_c t_c} = 0 \quad \forall \ j_a, t_a \quad \text{and} \quad S_T \in \mathcal{K}. \tag{13}$$

This equation relates the weight matrices of two spins in Eq. (12) and plays an important role in our exact renormalization-group method.

The third step of our method is to use Eq. (13) to carry out the renormalization-group process for the whole spin chain. Hereafter we use $T^{[i]}$ to denote the weight matrix at site i. Let us start the renormalization from the first two spins. By taking the first physical spin S as the input representation, we have the initial condition $T^{[1]}_{j_1} = 1$ with $j_1 = S$. According to Eqs. (12) and (13), we obtain $T^{[2]}_{j_1, j_2} = 1$. The output representation $j_2 \in \overline{\mathcal{K}}$, where $\overline{\mathcal{K}}$ is the orthogonal complement of $\mathcal{K}$. This simple test verifies the output states are the zero-energy local ground states of the projector Hamiltonian (5).

The renormalization group follows naturally as $T^{[2]} \to T^{[3]} \to \cdots$ under the renormalization condition in Eq. (13) and the constraints in Eqs. (8) and (9). Let us describe how to deal with these requirements simultaneously. Let us suppose that we already know the weight matrix $T^{[i-1]}$ and the goal is to calculate $T^{[i]}$. After taking the square of Eq. (13) and summing over j_a, t_a and $S_T \in \mathcal{K}$, we obtain

$$\sum_{j'_b t'_b} \sum_{j_b t_b} T^{[i]}_{j'_b t'_b, j_c t_c} \mathcal{M}^{[i]j_c}_{j'_b t'_b, j_b t_b} T^{[i]}_{j_b t_b, j_c t_c} = 0, \tag{14}$$

where the positive semidefinite real Hermitian matrix $\mathcal{M}^{[i]j_c}$ is given by

$$\mathcal{M}^{[i]j_c}_{j'_b t'_b, j_b t_b} = \sum_{S_T \in \mathcal{K}} \sum_{j_a t_a} T^{[i-1]}_{j_a t_a, j'_b t'_b} F^{j_a S j'_b}_{SS_T j_c} F^{j_a S j_b}_{SS_T j_c} T^{[i-1]}_{j_a t_a, j_b t_b}.$$

For every possible j_c from $j_b \otimes S$, we calculate the kernel of $\mathcal{M}^{[i]j_c}$, which give us the weight matrix $T^{[i]}$. If $\mathcal{M}^{[i]j_c}$ does not have kernel vectors satisfying Eq. (8), the corresponding output representation j_c must be discarded. If the kernel of $\mathcal{M}^{[i]j_c}$ has dimension larger than 1, the index t_c is used to tag the orthonormal kernel vectors for such j_c. Thus, the kernel vectors of $\mathcal{M}^{[i]j_c}$ constitute the columns of $T^{[i]}$ and the column indices j_c, t_c of $T^{[i]}_{j_b t_b, j_c t_c}$ denote the output representations. One can straightforwardly show that the resulting weight matrix $T^{[i]}$ satisfies the renormalization condition in Eq. (13) and the orthonormal constraint in Eq. (9), because $\mathcal{M}^{[i]j_c}$ is positive semidefinite and Hermitian.

B. Matrix-product states

In this section, we discuss the models which have a renormalization fixed point and then MPS as ground states. In our present exact renormalization scheme, the renormalization fixed point means that the output representations does not change when adding new spins and $T^{[i]}$ converges to a site-independent matrix.

Let us start by introducing two relevant concepts about MPS—injectivity and symmetry. We begin with the definition of injectivity[37]
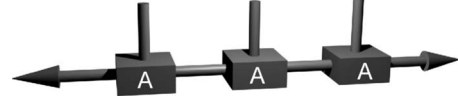

FIG. 3. Injectivity. A MPS is injective with injectivity length L_0 if for the MPS constructed for L_0 sites, different boundary conditions (linearly independent), represented by the cones in the figure, give rise to different states (linearly independent), and this does not happen for $L_0 - 1$ spins.

Definition 1 (Injectivity). Let $|\alpha\rangle \in \mathbb{C}^D$ be an orthonormal basis and $\{A^{[M]}\}^d_{M=1}$ the $D \times D$ Kraus operators defining a translationally invariant MPS. And let us consider the D^2 states for L sites defined as

$$|\psi^{(L)}_{\alpha\beta}\rangle = \sum_{M_1 \dots M_L} \langle \alpha | A^{[M_1]} \dots A^{[M_L]} | \beta \rangle | M_1 \dots M_L \rangle. \tag{15}$$

Then, we say that the MPS is injective (see Fig. 3) if there exists a finite L such that the vector space spanned by the vectors in Eq. (15) has dimension D^2. In other words, different boundary conditions turn into different states. The injectivity length L_0 is defined by the minimal number of sites for which injectivity is reached.

The interest of this definition comes from Refs. 5 and 7, where it is proven that injectivity is the necessary and sufficient condition for the existence of a parent Hamiltonian which has the MPS as a unique ground state with a nontrivial spectral gap above.

The other relevant result that we would like to recall here is the construction of translationally invariant MPS which are locally invariant under some symmetry group G. The following theorem provides the necessary and sufficient conditions[21] for that

Theorem 2 (Symmetry). Let $|\Psi\rangle \in (\mathbb{C}^d)^{\otimes N}$ be translationally invariant MPS defined by the Kraus operators $\{A^{[M]}\}^d_{M=1}$, and let u and U be two representations of a finite or a compact Lie group G. Then, $|\Psi\rangle$ is invariant under G in the sense of $u^{\otimes N}|\Psi\rangle = e^{i\theta N}|\Psi\rangle$ if and only if (see Fig. 4)

$$\sum_{M'} u_{M,M'} A^{[M']} = e^{i\theta} U A^{[M]} U^\dagger, \tag{16}$$

Here we call that u and U are the physical and virtual spin representations, respectively. Once these representations are fixed, the Kraus operators can be constructed by means of the Clebsch-Gordan coefficients together with a weight matrix. In our present SU(2) case, the Kraus operators are exactly given by the decomposition in Eq. (7).
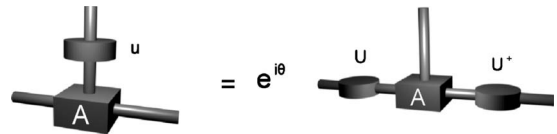

FIG. 4. Symmetry. The unitary u applied on the physical level is reflected in the virtual level as pair of unitaries U, as shown in Ref. 21.

TABLE I. Models with SU(2) invariance, nearest-neighbor interactions, and matrix-product ground states. L_0 is the injectivity length.

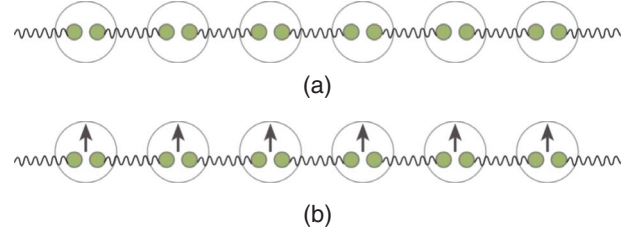| Spin	Set $\mathcal{K}$	Virtual spin	L_0
1	$\{2\}$	1/2	2
2	$\{3,4\}$	1	2
2	$\{2,4\}$	3/2	4
3	$\{4,5,6\}$	3/2	2
3	$\{2,4,6\}$	$0 \oplus 3$	6
4	$\{5,6,7,8\}$	2	2
4	$\{2,4,6,8\}$	$2 \oplus 5$	8
5	$\{6,7,8,9,10\}$	5/2	2
5	$\{2,4,6,8,10\}$	$5/2 \oplus 9/2 \oplus 15/2$	10
5	$\{3,7,8,9,10\}$	3	4

FIG. 5. (Color online) Ground-state physical picture. (a) The fixed-point-type MPS solutions have a VBS picture. Each dot denotes a virtual spin representation. The wavy lines represent the valence-bond singlets between virtual spins and the circles indicate the projection of two virtual spins onto physical spin representations. In the exact renormalization calculations, the fixed-point representations are from the tensor product of two virtual spins (edge states). (b) The partially ferromagnetic states have a magnetization plateau. The arrows denote a fully polarized virtual spin $(S-1)$ in a spin S partially ferromagnetic state.

Let us now remind the previous results about the MPS solutions for Hamiltonian (5). The best-known models belong to the AKLT family,[18] which are defined by $\mathcal{K}=\{S+1, S+2, \ldots, 2S\}$ for integer-spin S. The MPS of spin-S AKLT model have a VBS picture with irreducible virtual spin-$S/2$ representation. Another family of the models also have integer spin and the Hamiltonians are defined by $\mathcal{K}=\{2,4,\ldots,2S\}$,[32,33] which we call SO($2S+1$) symmetric family. For the $S=2$ model of this family, the MPS have irreducible virtual spin-3/2 representations,[32] which is equivalent to the SO(5) symmetric MPS in a two-leg electronic ladder.[34] For $S \geq 3$ cases, the properties of the corresponding MPS are less clear, even though their explicit wave functions were found.[32]

Now we turn to our results obtained by the exact renormalization group. For Hamiltonian (5), we check all possible $\mathcal{K}$ (Ref. 35) and then provide a complete list of fixed-point MPS solutions for $S \leq 4$, and a new solution for $S=5$. All these solutions are integer-spin models,[36] which are summarized in Table I. For $S \leq 4$, we conclude that there is no solution other than the above two families. For $S=5$, we find a new model, whose Hamiltonian is given by $\mathcal{K}=\{3,7,8,9,10\}$ and the ground state has a VBS picture with irreducible virtual spin-3 representations.

For the SO($2S+1$) family with $S \geq 3$, the exact renormalization group provides us a more comprehensible physical picture, which can be viewed as generalized VBS with *reducible* virtual spin representations. In Table I, we also listed the minimal number of blocked spins to reach the fixed-point representations. Since all these MPS are injective, this length scale is actually the injectivity length.[37]

Let us explain these results with an explicit example in the SO($2S+1$) family: the spin-3 model with $\mathcal{K}=\{2,4,6\}$. Through the exact renormalization group, we can observe that the output states reach the fixed-point representation $0 \oplus 0 \oplus 1 \oplus 2 \oplus 3 \oplus 3 \oplus 4 \oplus 5 \oplus 6$ after blocking six spins. To obtain the MPS, we do not really need to calculate the fixed-point Kraus operators by the renormalization group. According to Theorem 2, the fixed-point representations allow us to construct this MPS directly. For the present example, the fixed-point representations give an important hint

that the MPS has a VBS picture [see Fig. 5(a)] with SU(2) *reducible* virtual spin representation $0 \oplus 3$, which is quite different from the traditional VBS states with irreducible virtual spin representations, like AKLT states[18] or their extensions.[5]

With a chain beyond the injectivity length $L_0=6$, the tensor product of two $0 \oplus 3$ representations at the two boundaries yields the observed fixed-point representation in the renormalization group. For open boundary conditions, in thermodynamic limit, the unpaired representations $0 \oplus 3$ at the two edges are asymptotically free and become well-defined *edge states*. For periodic boundary conditions, all virtual spin representations are contracted into SU(2) singlets with neighboring sites and therefore the MPS is a global spin singlet.

The renormalization-group analysis has also been carried out for other models in SO($2S+1$) family. From Table I, one can see that, for $S \geq 3$, their matrix-product ground states have reducible virtual spin representations, which directly correspond to the edge states in an open chain. This provides more complete understanding of these systems. For all MPS in Table I, we present their explicit Kraus operators in Appendix.

Let us make a remark about these exactly solvable models. All their fixed-point MPS ground states have exponentially decaying correlations and there is an energy gap above the ground states, since they are injective. However, the different virtual spin representations (edge states) show that these MPSs belong to different quantum phases of matter. Therefore, once a new Hamiltonian $H=(1-x)H_1+xH_2$ is constructed from two solvable models H_1 and H_2 in Table I with the same spin S, at least one quantum phase transition is expected to occur when tuning x from 0 to 1. Since both MPS ground states for H_1 and H_2 preserve SU(2) symmetry, the local order parameter description breaks down and unconventional quantum phase transitions may emerge. Very recently, this idea has been exploited to study the possibility of a topological quantum phase transition in an $S=2$ chain.[38]

C. Partially ferromagnetic states

In this section, we discuss another class of models which do not have renormalization fixed points but still can be

solved exactly. The ground states of these models are partially ferromagnetic states.

This family includes both semi-integer spin models and integer-spin models. The Hamiltonian is defined by $\mathcal{K} = \{0, 1, \ldots, 2S-4, 2S\}$ and the physical spin $S \geq 5/2$. Their ground states are partially ferromagnetic states with a magnetization plateau $\langle S_i^z \rangle = S-1$. We also have found a physical picture [see Fig. 5(b)] for these states with partial magnetization. We prepare a spin-1 AKLT-type VBS state with virtual spin-1/2 and a spin-$(S-1)$ maximally polarized ferromagnetic state. In each site, we recover the physical spin-S Hilbert space by $(S-1) \otimes 1 \to S$, which is achieved by applying local projections.

Let us consider a typical example—the spin-5/2 model with $\mathcal{K} = \{0, 1, 5\}$. For a block of N_0 spins, the AKLT part contributes representations $0 \oplus 1$ and the polarized ferromagnetic part contributes representation $3N_0/2$. Thus, the total spin of the N_0-spin block is given by the tensor product of representations from these two parts

$$(0 \oplus 1) \otimes \frac{3N_0}{2} = \left(\frac{3N_0}{2} - 1\right) \oplus \frac{3N_0}{2} \oplus \frac{3N_0}{2} \oplus \left(\frac{3N_0}{2} + 1\right).$$
(17)

For two spins ($N_0 = 2$), the allowed representations are $2 \oplus 3 \oplus 3 \oplus 4$ and cannot reach $\mathcal{K} = \{0, 1, 5\}$, which means that the partially ferromagnetic state is the zero-energy ground state of the projector Hamiltonian. In the exact renormalization process, we found that the four output representations in Eq. (17) are the only output representations for $N_0 \geq 6$. By adding one additional spin, the total spin of the four representations is increased by 3/2. These observations actually strongly suggest the partially ferromagnetic picture of the ground state.

One may ask why this class starts with $S = 5/2$ rather than $S = 2$. The reason is the following. For spin-2 model $\mathcal{K} = \{0, 4\}$, the renormalization group shows that the number of output representations does not saturate, which means that the partially ferromagnetic state is not the only ground state of the Hamiltonian.

Compared to the fixed-point MPS solutions in Sec. III B, the partially ferromagnetic states have a long-range order and thus break the SU(2) symmetry. According to Goldstone theorem, we expect gapless spin-wave excitations above the ground state, which is quite different from the gapped fixed-point MPS with exponentially decaying correlations.

IV. CONCLUSIONS AND PERSPECTIVES

We have introduced a real-space exact renormalization group adapted to the SU(2) symmetry, which is well suited for finding exactly solvable quantum spin Hamiltonians with nearest-neighbor interactions.

The list of solutions can be divided into two classes according to the renormalization-group behavior. In the first class, the models are quantum integer-spin chains with renormalization fixed points and matrix-product ground states. For $S \leq 4$, we show that the AKLT family and the SO($2S+1$) family exhaust all possible solutions. In the SO($2S+1$) fam-

ily, the renormalization group provides a natural explanation for the edge states of the MPS by providing a generalized VBS picture with reducible virtual spin representation. Furthermore, we obtain a new solvable model for $S = 5$ beyond the existing families. In the second class, the models have partially ferromagnetic ground states with a magnetization plateau. This solvable family exists for $S \geq 5/2$ and contains both integer spin and semi-integer spin models. The partially ferromagnetic ground states have gapless spin-wave excitations, which are quite different from the gapped MPS in the first class.

Beyond the present work, it would be quite interesting to generalize the method to spin chains beyond nearest-neighbor interactions and models in higher dimensions, especially adapted to PEPS formalism. Furthermore, the method may be used to explain an open question by Östlund and Rommer[10] about which representations must be introduced in the renormalization group with SU(2) symmetry.

Finally, we also expect a natural extension of the present exact renormalization formalism to quantum spin chains with other symmetry groups.[39]

APPENDIX: KRAUS OPERATORS OF THE MATRIX-PRODUCT STATES

In this appendix, we explicitly present the Kraus operators needed for the definition of the MPS in Table I. As we mentioned, the Kraus operators with SU(2) symmetry are parametrized by Eq. (7), which requires both the set $\mathcal{V}$ containing the SU(2) virtual spin representations and the weight matrix T.

For irreducible virtual spin representations, the set $\mathcal{V}$ contains a single representation j_a and therefore $T = 1$. In this case, the Kraus operators are simply the Clebsch-Gordan coefficients

$$A^{[S,M]}_{j_a m_a, j_a m_b} = \langle j_a m_a, SM | j_a m_b \rangle.$$
(A1)

For reducible virtual spin representations, the set $\mathcal{V}$ has multiple SU(2) representations and the weight matrix T is necessary. The Kraus operators are given by

$$A^{[S,M]}_{j_a m_a, j_b m_b} = T_{j_a, j_b} \langle j_a m_a, SM | j_b m_b \rangle,$$
(A2)

where the index t is suppressed because no degeneracy occurs in $\mathcal{V}$ for our models. We use a convention to define the matrix T such that the row and the column indices j_a, j_b are arranged in an incremental order. For instance, the $S=3$ model with $\mathcal{K} = \{2,4,6\}$ has virtual representation $0 \oplus 3$ and

$$T = \begin{pmatrix} T_{0,0} & T_{0,3} \\ T_{3,0} & T_{3,3} \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{-1}{\sqrt{7}} \\ 1 & \sqrt{\dfrac{6}{7}} \end{pmatrix}. \tag{A3}$$

For the $S=4$ model with $\mathcal{K} = \{2,4,6,8\}$, we have virtual representation $2 \oplus 5$ and

$$T = \begin{pmatrix} \dfrac{1}{3}\sqrt{\dfrac{7}{2}} & \dfrac{1}{3}\sqrt{\dfrac{5}{2}} \\ \dfrac{-1}{3}\sqrt{\dfrac{11}{2}} & \dfrac{1}{3}\sqrt{\dfrac{13}{2}} \end{pmatrix}. \tag{A4}$$

For the $S=5$ model with $\mathcal{K} = \{2,4,6,8,10\}$, we have virtu-al representation $5/2 \oplus 9/2 \oplus 15/2$ and

$$T = \begin{pmatrix} \dfrac{1}{11}\sqrt{\dfrac{21}{2}} & \dfrac{-3}{\sqrt{22}} & \dfrac{-1}{11}\sqrt{\dfrac{21}{2}} \\ -\sqrt{\dfrac{15}{22}} & \sqrt{\dfrac{3}{26}} & -\sqrt{\dfrac{85}{286}} \\ \dfrac{2\sqrt{7}}{11} & 2\sqrt{\dfrac{17}{143}} & \dfrac{1}{11}\sqrt{\dfrac{969}{13}} \end{pmatrix}. \tag{A5}$$

It is straightforward to show that these solutions satisfy Eqs. (8), (9), and (13).

[1] For recent reviews, see J. I. Cirac and F. Verstraete, J. Phys. A **42**, 504004 (2009); F. Verstraete, V. Murg, and J. I. Cirac, Adv. Phys. **57**, 143 (2008).

[2] F. Verstraete and J. I. Cirac, Phys. Rev. B **73**, 094423 (2006).

[3] M. B. Hastings, J. Stat. Mech.: Theory Exp. (2007), P08024.

[4] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010).

[5] M. Fannes, B. Nachtergaele, and R. F. Werner, Commun. Math. Phys. **144**, 443 (1992).

[6] A. Klümper, A. Schadschneider, and J. Zittartz, J. Phys. A **24**, L955 (1991); Z. Phys. B: Condens. Matter **87**, 281 (1992).

[7] D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac, Quantum Inf. Comput. **7**, 401 (2007).

[8] S. R. White, Phys. Rev. Lett. **69**, 2863 (1992).

[9] U. Schollwöck, Rev. Mod. Phys. **77**, 259 (2005).

[10] S. Östlund and S. Rommer, Phys. Rev. Lett. **75**, 3537 (1995).

[11] F. Verstraete, D. Porras, and J. I. Cirac, Phys. Rev. Lett. **93**, 227205 (2004).

[12] G. Vidal, Phys. Rev. Lett. **93**, 040502 (2004).

[13] J. I. Cirac and G. Sierra, Phys. Rev. B **81**, 104431 (2010).

[14] F. Verstraete and J. I. Cirac, Phys. Rev. Lett. **104**, 190405 (2010).

[15] F. Verstraete and J. Cirac, arXiv:cond-mat/0407066 (unpublished).

[16] G. Sierra and M. Martín-Delgado, *Proceedings of the Workshop on the Exact Renormalization Group*, Faro, Portugal, 10–12 September, 1998 (World Scientific, Singapore); arXiv:cond-mat/9811170.

[17] T. Nishino, Y. Hieida, K. Okunishi, N. Maeshima, Y. Akutsu, and A. Gendiar, Prog. Theor. Phys. **105**, 409 (2001).

[18] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987).

[19] F. D. M. Haldane, Phys. Lett. A **93**, 464 (1983); Phys. Rev. Lett. **50**, 1153 (1983).

[20] D. P. Arovas, A. Auerbach, and F. D. M. Haldane, Phys. Rev. Lett. **60**, 531 (1988).

[21] M. Sanz, M. M. Wolf, D. Pérez-García, and J. I. Cirac, Phys. Rev. A **79**, 042308 (2009).

[22] N. Schuch, I. Cirac, and F. Verstraete, Phys. Rev. Lett. **100**, 250501 (2008).

[23] S. Bravyi, arXiv:quant-ph/0602108 (unpublished).

[24] R. Movassagh, E. Farhi, J. Goldstone, D. Nagaj, T. J. Osborne, and P. W. Shor, Phys. Rev. A **82**, 012318 (2010).

[25] N. de Beaudrap, M. Ohliger, T. Osborne, and J. Eisert, Phys. Rev. Lett. **105**, 060504 (2010).

[26] C. R. Laumann, R. Moessner, A. Scardicchio, and S. L. Sondhi, Quantum Inf. Comput. **10**, 0001 (2010); C. R. Laumann, A. M. Läuchli, R. Moessner, A. Scardicchio, and S. L. Sondhi, Phys. Rev. A **81**, 062345 (2010).

[27] T. Kennedy, J. Phys. A **25**, 2809 (1992).

[28] J. Dukelsky, M. A. Martín-Delgado, T. Nishino, and G. Sierra, Europhys. Lett. **43**, 457 (1998).

[29] I. McCulloch and M. Gulacsi, Europhys. Lett. **57**, 852 (2002).

[30] S. Singh, H.-Q. Zhou, and G. Vidal, New J. Phys. **12**, 033029 (2010); S. Singh, R. Pfeifer, and G. Vidal, arXiv:0907.2994 (unpublished).

[31] D. M. Brink and G. R. Satchler, *Angular Momentum* (Clarendon Press, Oxford, 1968).

[32] H.-H. Tu, G.-M. Zhang, and T. Xiang, Phys. Rev. B **78**, 094404 (2008); J. Phys. A **41**, 415201 (2008).

[33] H.-H. Tu, G.-M. Zhang, T. Xiang, Z.-X. Liu, and T.-K. Ng, Phys. Rev. B **80**, 014401 (2009).

[34] D. Scalapino, S.-C. Zhang, and W. Hanke, Phys. Rev. B **58**, 443 (1998).

[35] In practice, one can rule out some models to simplify the calculation. For example, the element $2S$ is always included in $\mathcal{K}$, otherwise the model has (at least) a fully polarized ferromagnetic ground state. We also use the sets $\mathcal{K}'$ for AKLT models and SO($2S+1$) models. Since these known models have unique ground states, there is no need to check the sets $\mathcal{K}$ satisfying $\mathcal{K} \subset \mathcal{K}'$ or $\mathcal{K} \supset \mathcal{K}'$.

[36] For semi-integer spin chains, it is not possible to have a translationally invariant MPS as a unique ground state. According to the Lieb-Schultz-Mattis theorem, SU(2) symmetric spin chains with semi-integer spins are critical if the ground state is unique. On the contrary, the fixed-point MPS should have an energy gap and exponentially decaying correlations. However, the possibility of the MPS solution with breaking translational symmetry can not be ruled out. Although these models are under the scope of exact renormalization method, we do not obtain any such solution for $S \leq 7/2$.

[37] M. Sanz, D. Pérez-García, M. Wolf, and J. Cirac, arXiv:0909.5347 (unpublished).

[38] D. Zheng, G. Zhang, T. Xiang, and D. Lee, arXiv:1002.0171 (unpublished); J. Zang, H.-C. Jiang, Z.-Y. Weng, and S.-C. Zhang, Phys. Rev. B **81**, 224430 (2010).

[39] M. Greiter and S. Rachel, Phys. Rev. B **75**, 184441 (2007); D. Schuricht and S. Rachel, *ibid.* **78**, 014430 (2008); D. P. Arovas, K. Hasebe, X.-L. Qi, and S.-C. Zhang, *ibid.* **79**, 224404 (2009); K. Motegi, Phys. Lett. A **374**, 3112 (2010).